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Approximation of Integrals via Monte Carlo Methods, with an Application to Calculating Radar Detection Probabilities

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ABSTRACT

The approximation of definite integrals using Monte Carlo simulations is the focus of the work presented here. The general methodology of estimation by sampling is introduced, and is applied to the approximation of two special functions of mathematics: the Gamma and Beta functions. A significant application, in the context of radar detection theory, is based upon the work of [Shnidman 1998]. The latter considers problems associated with the optimal choice of binary integration parameters. We apply the techniques of Monte Carlo simulation to estimate binary integration detection probabilities.

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EXECUTIVE SUMMARY

The performance analysis of a radar detection scheme requires estimation of probabilities of false alarm and detection, under various clutter scenarios. These probabilities, which often appear as definite integrals, are frequently analytically difficult to evaluate. Hence, numerical approximation schemes are employed. Monte Carlo estimators use statistical simulation to evaluate such integrals. As with any approximation scheme, there are limitations and drawbacks in its application. One of the major difficulties with Monte Carlo estimators is that very large sample sizes may be required, in order to achieve a reasonable estimate. This is especially true in the context of estimating probabilities of rare events, such as radar false alarms.

The purpose of this report is to examine the Monte Carlo estimation of integrals in general. After formulating the scheme, applications to the evaluation of two special functions are considered. The success of an estimator will be decided on its performance in terms of providing a reasonable estimate for the smallest sample size possible.

The major application in this report will be to obtain estimates for a detection probability in a binary integration context. Under the assumption of a Swerling target model, an expression for the binary integrated probability of detection is obtained in Shnidman's 1998 paper entitled *Binary integration for Swerling target fluctuations* (IEEE Transactions on Aerospace and Electronic Systems, Volume 34, pp. 1043-1053). We apply Monte Carlo simulations, together with some functional approximations, to estimate this probability for Swerling 1 and 3 target models.

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Graham V. Weinberg was educated at The University of Melbourne, graduating in 1996 with a Bachelor of Science degree with Honours in Mathematics and Statistics. Following this he was conferred in 2001 with a Doctor of Philosophy degree in Applied Probability, also at The University of Melbourne. His doctoral thesis examined probability approximations for random variables and point processes, using a technique known as the Stein–Chen Method. The latter enables a probabilistic assessment of the errors in distributional approximations of stochastic processes.

His post-doctoral work began with a 6 month contractual position with a finance group known as Webmind, investigating artificial intelligence applied to stock market prediction. This was based in the Department of Psychology at The University of Melbourne. Following this was employment as a Research Fellow in the Teletraffic Research Centre, at the University of Adelaide. The latter involved working as a telecommunications consultant to business and industry. Since early 2002 he has been employed as a Research Scientist with DSTO, examining radar performance analysis techniques for the P3 Orion maritime patrol aircraft. From mid 2004 he has changed directions in terms of research, now working on problems related to radar resource management and waveform analysis.

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Ross Kyprianou graduated from the University of Adelaide in 1985 with a Bachelor of Science degree in Pure Mathematics and Mathematical Physics. He went on to complete a Post Graduate Diploma in Computer Science in 1987. After a short period teaching secondary students in mathematics and science, he worked in various government positions in a number of diverse roles including I.T. management, software security, system and database administration and software design and implementation.

He joined DSTO in 1990, where he has been part of research efforts in the areas of target radar cross section, digital signal processing, inverse synthetic aperture radar and radar detection using both software modelling and mathematical analysis and techniques.

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1 Introduction

It is a common occurrence, in the study of radar detection performance, to find it analytically intractable to construct a closed form expression for probabilities of interest. The two key performance measures, the probability of false alarm and probability of detection, used in the analysis of Constant False Alarm Rate (CFAR) detectors, typically involve integrals that cannot be readily evaluated. Monte Carlo methods are thus often used, and in the context of false alarm probabilities, much work has been generated on the construction of suitable suboptimal biasing densities (see [Weinberg 2004] and references contained therein). In a CFAR system, the false alarm probability is set to a very small number, and consequently Monte Carlo estimators of this may need a very large sample size to achieve a nonzero estimate.

In the context of CFAR detection, we are testing whether a test observation x_0 represents a target or not. This decision is based on whether this observation exceeds a weighted “average” measure of the clutter level. Mathematically, if Ξ_0 is the random variable representing the test observation, and $\Xi_1, \Xi_2, \dots, \Xi_n$ are clutter statistics, we declare a detection if $\Xi_0 > \frac{\tau}{n} \psi(\Xi_1, \Xi_2, \dots, \Xi_n)$, where τ is the threshold, and ψ is a function which measures the clutter level. The probability of false alarm and detection can be written in the form

$$P = \int_{\Gamma} \int_{\Gamma} \cdots \int_{\Gamma} \zeta(x_0, x_1, \dots, x_n) \xi(x_0, x_1, \dots, x_n) dx_0 dx_1 \cdots dx_n. \quad (1)$$

Here $\zeta(x_0, x_1, \dots, x_n) = I[x_0 > \frac{\tau}{n} \psi(x_1, x_2, \dots, x_n)]$, where I is defined by

$$I[x \in A] = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

The term involving ξ is the joint density of the cell under test and the clutter statistics. Whether the integral (1) is for a detection or false alarm probability will depend on the distribution of the cell under test statistic. In either direction, what becomes apparent is that this integral will often be quite difficult to evaluate analytically. Hence a numerical approximation scheme is required. Due to the presence of a density in the integral (1), Monte Carlo estimation seems to be a natural choice.

The objective of this report is to illustrate the application of Monte Carlo methods to the more general problem of integral evaluation. The ideas of changing a simulation distribution, also known as *Importance Sampling*, will be illustrated in this context. Monte Carlo techniques will be illustrated in the application to evaluation of some special functions that arise in mathematics. A specific radar related application appears in the evaluation of a detection probability integral. The latter arises in the context of binary integration with Swerling target models, and appears in [Shnidman 1998]. A Monte Carlo scheme is used, as well as some other approximations, to construct estimates for the binary integrated detection probability for two Swerling models.

We begin by introducing the basic ideas of Monte Carlo methods.

2 Monte Carlo Techniques

The application of simulation methods to the estimation of difficult integrals began with the work of [Kahn 1950] and others working in nuclear physics in the 1940s. An early radar application to estimation of false alarm probabilities is [Mitchell 1981]. The basis for Monte Carlo techniques is the *Strong Law of Large Numbers* (SLLN) (see [Billingsley 1986]). This states that a series of independent and identically distributed (IID) random variables, normalised by the number of terms in the series, will converge to the mean of any one of the terms in the series. Mathematically, this implies that if $\Xi_1, \Xi_2, \dots, \Xi_m$ is a sequence of IID random variables with finite mean $E[\Xi]$, then

$$\lim_{m \rightarrow \infty} \frac{\sum_{j=1}^m \Xi_j}{m} = E[\Xi], \quad (2)$$

except on a set of probability measure zero. This suggests that, for sufficiently large m , the average of the random variables in (2) can be approximated by its mean. Where this applies, in the context of interest, is that it enables the evaluation of integrals through *simulation*.

Consider the integral $I = \int_{\Omega} \omega(x) \xi(x) dx$, where ξ is a density on Ω , and ω is a function. This integral is a statistical mean of a random variable Ξ on Ω with density ξ . Hence we can write $I = E[\omega(\Xi)]$. Now if $\Xi_1, \Xi_2, \dots, \Xi_m$ is an IID sequence of random variables, then the SLLN (2) implies that

$$\lim_{m \rightarrow \infty} \frac{\sum_{j=1}^m \omega(\Xi_j)}{m} = E[\omega(\Xi)] = I, \quad (3)$$

except on a set of probability measure zero. Hence, by applying (3), we can deduce that

$$I = \int_{\Omega} \omega(x) \xi(x) dx \approx \frac{\sum_{j=1}^m \omega(z_j)}{m}, \quad (4)$$

where the sequence z_1, z_2, \dots, z_m consists of realisations of the variables $\Xi_1, \Xi_2, \dots, \Xi_m$. The result in (4) implies that the integral I can be estimated by generating a series of realisations of a random variable with density ξ , and evaluating the average of the function ω over these realisations. This is a computationally simple exercise in theory. As remarked previously, an underlying problem with Monte Carlo methods is that it may require a very large sample size to achieve a reasonable estimate. Changing the biasing density can sometimes rectify this, and this will be considered in the discussion to follow.

In cases where we have a definite integral of a function that is not a density on the integral's domain, it is still possible to apply Monte Carlo methods. To illustrate this, suppose I is a definite integral of the form

$$I = \int_{\mathcal{A}} \zeta(x) dx, \quad (5)$$

where $\zeta : \mathcal{A} \rightarrow \mathbf{R}$ is a continuous nonnegative real valued function on the interval $\mathcal{A} = [a, b]$. We do not assume that the boundaries of this interval are finite, so that we allow for integrals on infinite domains. We do assume, however, that the integral exists, in a Riemann or Lebesgue sense. We would like to apply the SLLN to (5), in order to apply a Monte Carlo approximation. We can construct a probability density function ξ on \mathcal{A} , and modify the integral to

$$I = \int_{\mathcal{A}} \frac{\zeta(x)}{\xi(x)} \xi(x) dx = \mathbb{E}\omega(\Xi), \quad (6)$$

where $\omega(x) = \frac{\zeta(x)}{\xi(x)}$, and the expectation in (6) is with respect to a random variable Ξ with density ξ . We require this density to be nonzero so that $\omega(x)$ is well defined. The existence of such densities can be shown by considering the four types of integral domains. If the integral's domain is an interval of the form $[a, b]$, where both a and b are finite, then one can choose a uniform distribution over this domain. In the case of an integral with domain $[a, \infty)$ or $(-\infty, b]$, where both a and b are finite, an Exponential distribution can be used. The final case, where the integral is over the whole real line, a Gaussian distribution can be used. This procedure is often referred to as *Importance Sampling*. [Weinberg 2004] contains a detailed list of Importance Sampling references.

An application of the SLLN to (6) results in the Monte Carlo estimator

$$\widehat{I}_N = \frac{1}{N} \sum_{j=1}^N \omega(\Xi_j), \quad (7)$$

where each $\Xi_j \stackrel{d}{=} \Xi$. In general, the expression $\Phi \stackrel{d}{=} \Psi$ means that the distributions of random variables Φ and Ψ are equal, so that for every set \mathcal{A} in a common domain, $P(\Phi \in \mathcal{A}) = P(\Psi \in \mathcal{A})$. Estimator (7) is an unbiased estimator of I , since

$$\mathbb{E}\widehat{I}_N = \mathbb{E}\omega(\Xi) = \int_{\mathcal{A}} \omega(x)\xi(x)dx = I. \quad (8)$$

Thus estimates are centred on the integral being approximated. The variance of (7) can be shown to be

$$\begin{aligned} \mathbb{V}\widehat{I}_N &= \frac{1}{N} \left(\mathbb{E}(\omega(\Xi)^2) - (\mathbb{E}(\omega(\Xi)))^2 \right) \\ &= \frac{1}{N} \left(\int_{\mathcal{A}} \omega^2(x)\xi(x)dx - I^2 \right) \\ &= \frac{1}{N} \left(\int_{\mathcal{A}} \frac{\zeta^2(x)}{\xi(x)} dx - I^2 \right). \end{aligned} \quad (9)$$

We would like to have an estimator whose variance is as small as possible. Notice that with the choice of $\xi(x) = \frac{\zeta(x)}{I}$ on \mathcal{A} , the variance (9) reduces to zero. This biasing density is known as *the optimal solution*, but is of no practical use because it depends on the quantity being estimated. Many authors have used knowledge of the optimal solution to construct a suboptimal biasing density, with varying degrees of success (see [Gerlack 1999] and [Orsak and Aazhang 1989]). Its form suggests that a suitable biasing density should be concentrated on the integral's domain, and in some sense proportional to the integrand.

Thus, if we are presented with an integral of the form (5), there are three ways to proceed in terms of the Monte Carlo approach. If integrand $\zeta(x)$ contains a density on the integral's domain, we can use this to simulate. If this is not the case, then we can insert a biasing density, and alter the integrand. The other possibility is that we can change biasing densities even if ζ does contain a density.

We now turn to the issue of a suitable choice of biasing density. There are many different distributions from which one can select a biasing distribution. The Weibull family of distributions, $\mathcal{W}(\alpha, \beta)$, with nonnegative parameters α and β , has probability density function

$$\xi_{\mathcal{W}}(x) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}},$$

and can be simulated via

$$\Phi_{\mathcal{W}}^{-1}(R) = \beta (-\log(R))^{\frac{1}{\alpha}} \stackrel{d}{=} \mathcal{W}(\alpha, \beta),$$

where $\Phi_{\mathcal{W}}(x)$ is the cumulative distribution function, and $R \stackrel{d}{=} R(0, 1)$ is a continuous uniform distribution on the unit interval. There are a number of important special cases of the Weibull distribution. Observe that $\mathcal{W}(1, \beta)$ is an Exponential distribution, while $\mathcal{W}(2, \beta)$ is Rayleigh.

The Gamma family of distributions, $\mathcal{G}(r, \beta)$, has density

$$\xi_{\mathcal{G}}(x) = \frac{\left(\frac{x}{\beta}\right)^{r-1} e^{-\frac{x}{\beta}}}{\beta \gamma(r)},$$

where $\gamma(r)$ is a normalising constant, called the Gamma function, and we also assume parameters r and β are nonnegative. It is analytically impossible to write down the inverse of the cumulative distribution function of the Gamma distribution. However, it is possible to simulate from this distribution, using the fact that if Ξ is a random variable with the Gamma distribution with parameters r and β , then $\Xi = \Xi_1 + \Xi_2 + \dots + \Xi_r$, where each Ξ_j is an IID Exponential random variable with parameter β . Thus realisations of a Gamma distribution can be obtained by summing independent realisations from Exponential distributions.

The success of a biasing density is largely application dependent. In the next section we will investigate the result of applying different biasing densities to the Monte Carlo estimation of some special functions.

3 Monte Carlo Approximations of some Special Functions

In order to illustrate the application of Monte Carlo simulation to the evaluation of integrals, we examine two special functions. The Gamma and Beta functions arise in a number of contexts in mathematics and statistics. In applied mathematics, these functions appear in the theory of Bessel functions [Bowman 1958]. They arise as normalising constants in the definition of certain distributions in the theory of probability [Ross 1983]. The Gamma function $\gamma(n) : \mathbf{R}^+ \rightarrow \mathbf{R}$ is defined by the integral

$$\gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \quad (10)$$

and the Beta function $\beta(m, n) : \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}$ is

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx. \quad (11)$$

For nonnegative integral n , it can be shown that $\gamma(n+1) = n!$. The Beta function can be decomposed into an expression involving Gamma functions. Specifically, it can be shown that $\beta(m, n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$. Except for some simple cases, it is necessary to use numerical methods to evaluate these integrals in practice. We will apply Monte Carlo techniques to both (10) and (11). In the case of (10), the exponential term $\xi(x) := e^{-x}$ is a density on the interval $[0, \infty)$. Hence, an appropriate Monte Carlo estimator of (10) is

$$\widehat{\gamma}_1(n, N) = \frac{1}{N} \sum_{j=1}^N \Xi_j^{n-1}, \quad (12)$$

where each Ξ is a random variable generated from a distribution with density ξ . In this case, since ξ is the density of an Exponential random variable with parameter 1, we can use $\Xi \stackrel{d}{=} -\log(R)$ to simulate the distribution, where R is a uniform density on the interval $[0, 1]$.

We can construct an alternative estimator by simulating from a distribution with density from the Weibull class. If ξ is the Weibull density $\xi(x) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha}$, then an estimator based on this is

$$\begin{aligned} \widehat{\gamma}_2(n, N) &= \frac{1}{N} \sum_{j=1}^N \frac{\Xi_j^{n-1} e^{-\Xi_j}}{\xi(\Xi_j)} \\ &= \frac{\beta^\alpha}{\alpha N} \sum_{j=1}^N \Xi_j^{n-\alpha} e^{-\Xi_j + \left(\frac{\Xi_j}{\beta}\right)^\alpha}, \end{aligned} \quad (13)$$

where each $\Xi_j = \beta(-\log R_j)^{\frac{1}{\alpha}}$, with $R_j \stackrel{d}{=} R[0, 1]$.

Figures 1-3 in Appendix A contain simulations to estimate the Gamma function, using both the estimators (12) and (13). The *Matlab* code used to generate the estimates can be

found in Appendix B. Each plot compares these two estimators to the exact value for the Gamma function, or a numerical approximation based upon an inbuilt *Matlab* function. Figures 1 and 2 are based upon a sample size of $N = 10,000$, while that of Figure 3 is based upon a sample of size of $N = 100,000$. The Weibull estimator (13) was chosen with parameters $\alpha = 1$ and $\beta = 10$. It was found empirically that other values for α did not provide good estimates. The simulations show there is reasonably good convergence to the exact value for approximately 100,000 simulation runs.

The Beta function can also be estimated using Monte Carlo estimators. The difference is that the domain of the integral is the unit interval $[0, 1]$. A suitable density on this interval is the standard uniform one, which is $\xi(x) = 1$, for $x \in [0, 1]$. In this case, the Monte Carlo estimator is

$$\widehat{\beta}_1(m, n, N) = \frac{1}{N} \sum_{j=1}^N \Xi_j^{m-1} (1 - \Xi_j)^{n-1}, \quad (14)$$

where in this case, each Ξ_j is a random variable with the standard uniform density on $[0, 1]$. There are also some other natural choices for biasing densities for the Beta function. We can choose $\xi(x) = mx^{m-1}$, and introduce a scaling factor of m to the Beta function's integral. Hence the estimator is

$$\widehat{\beta}_2(m, n, N) = \frac{1}{mN} \sum_{j=1}^N (1 - \Xi_j)^{n-1}, \quad (15)$$

where Ξ is a random variable with the density ξ . This can be simulated from the fact that $R^{\frac{1}{m}} \stackrel{d}{=} \Xi$, where $R \stackrel{d}{=} R[0, 1]$ is a uniform distribution on the unit interval. Note that a biasing density can also be based on the second term in the integrand. Specifically, we could make the choice of $\xi(x) = n(1 - x)^{n-1}$, which is also a density on the unit interval.

As remarked in Section 2, it is possible to simulate from any distribution we can define on the integral's domain. In the current context, we could insert a modified Exponential distribution into the Beta integral, and use this for simulation. To illustrate, note that the function $\xi(x) = \frac{e^{-x}}{1-e^{-1}}$ is a density on $[0, 1]$, and a random variable Ξ with this as its density can be simulated by $-\log(1 - (1 - e^{-1})R) \stackrel{d}{=} \Xi$, where as before R is the standard uniform distribution on the unit interval. This distribution is referred to as a truncated Exponential distribution. Thus a third estimator of the Beta function is

$$\widehat{\beta}_3(m, n, N) = \frac{1 - e^{-1}}{N} \sum_{j=1}^N e^{\Xi_j} \Xi_j^{m-1} (1 - \Xi_j)^{n-1}, \quad (16)$$

where Ξ_j is generated from the truncated Exponential distribution.

Figures 4-6 in Appendix A contain a number of simulations of the three estimators (14)-(16). For given values of m and n , these estimators are compared to the exact result. In contrast to the estimates for the Gamma function, there is rapid convergence in this case. After only 100 simulations, the estimators are giving very good estimates of the Beta function.

4 Monte Carlo Approximations of Detection Probabilities

We now turn to the problem of estimating radar detection probabilities using Monte Carlo methods. The detection probability under consideration arises in the context of binary integration for Swerling target fluctuations [Shnidman 1998]. As an alternative to coherent integration of N pulses [Levanon 1988], binary integration determines how many single pulse exceedences of the threshold occur. A target detection is declared if there are at least M such exceedences, for a prescribed value of M , within N observations. A key issue is to determine the optimal value of M , which is the focus of [Shnidman 1998]. We will instead assume such an optimal value has been determined, and will investigate the corresponding detection probability integrals.

The following is taken from [Shnidman 1998], with slight modification of notation. Define the cumulative distribution function of Binomial probabilities between two integers N and M , with $1 \leq M \leq N$, to be

$$E(N, M, p) = \sum_{k=M}^N \binom{N}{k} (1-p)^{N-k} p^k, \quad (17)$$

for some $0 \leq p \leq 1$. The binary false alarm probability is $P_{FA} = E(N, M, p_1)$, where p_1 is the single pulse false alarm probability. The single pulse normalised threshold is $\tau = -\log(p_1)$. We define Ξ to be the target normalised signal to noise ratio (SNR), which under Swerling models, we assume has a Gamma distribution with parameters $r = l$ and $\beta = \frac{\xi_0}{l}$. The parameter l is a fluctuation parameter, and ξ_0 is the average normalised SNR. The density of Ξ is thus

$$\xi(t|\xi_0, l) = \frac{t^{l-1}}{\gamma(l)} \left(\frac{l}{\xi_0} \right)^l e^{-\frac{lt}{\xi_0}}. \quad (18)$$

We let $p_c(t, \tau)$ be the single pulse probability of detection, for a constant target with single pulse normalised SNR level t . Then

$$\begin{aligned} p_c(t, \tau) &= \int_{\tau}^{\infty} e^{-(\nu+t)} I_0(2\sqrt{\nu t}) d\nu \\ &= e^{-(\tau+t)} \int_0^{\infty} e^{-\nu} I_0(2\sqrt{(\nu+\tau)t}) d\nu, \end{aligned} \quad (19)$$

where I_0 is the modified Bessel function of order zero [Bowman 1958]. For the four Swerling target models considered in [Shnidman 1998], the binary integrated probability of detection turns out to be

$$P_D = \int_0^{\infty} E(N, M, p(t, \tau)) \xi(t|\xi_0, l) dt, \quad (20)$$

where the form of $p(t, \tau)$ depends on the Swerling case. As explained in [Shnidman 1998], for Swerling 1 and 3, $p(t, \tau) = p_c(t, \tau)$. In the case of Swerling 2,

$$p(t, \tau) = \int_0^{\infty} p_c(\rho, \tau) \xi(\rho|\xi_0, l=1) d\rho = e^{-\frac{\tau}{1+\xi_0}}, \quad (21)$$

where the latter equality can be demonstrated analytically. It is important to note that the fluctuation parameter l is taken to be 1, for Swerling 1 and 2 target models, and is assumed to be 2 for Swerling cases 3 and 4. In the case of Swerling 4,

$$p(t, \tau) = \int_0^\infty p_c(\rho, \tau) \xi(\rho | \xi_0, l = 2) d\rho, \quad (22)$$

the difference between this and (21) being the difference in fluctuation parameters. When applied to (20), the Swerling 2 and 4 expressions for $p(t, \tau)$ do not depend on t , and so (20) reduces to the cumulative sum of binomial probabilities (17). Hence we do not examine these cases further, since Monte Carlo simulations are not required. We would like to obtain estimates of (20), using a Monte Carlo approximations, for Swerling 1 and 3 target models.

The presence of the Gamma density in (20) suggests that this could be used as a simulation density. Hence, a Monte Carlo estimator for the detection probability (20) is

$$\hat{P}_D(N, M, K, \tau) = \frac{1}{K} \sum_{j=1}^K E(N, M, p_c(\Xi_j, \tau)), \quad (23)$$

where each Ξ_j is a Gamma random variable with density given by (18). The Gamma variables can be simulated by adding $r = l$ independent realisations of Exponential random variables with parameter $\beta = \frac{\xi_0}{l}$.

We now need to approximate $p_c(t, \tau)$. There are two possibilities. Firstly, from [Shnidman 1995] and [Weinberg and Kyprianou 2005], we have

$$\begin{aligned} p_c(t, \tau) &= e^{-(\tau+t)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=0}^k \frac{\tau^j}{j!} \\ &= e^{-(\tau+t)} \times \\ &\quad 1 + t(1 + \tau) + \frac{t^2}{2!} \left(1 + \tau + \frac{\tau^2}{2!} \right) + \frac{t^3}{3!} \left(1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} \right) + \dots, \end{aligned} \quad (24)$$

which yields the approximation

$$\log(p_c(t, \tau)) \approx -\tau + \tau t - \left(\tau + \frac{\tau^2}{2} \right)^2 \frac{t^2}{2!}. \quad (25)$$

Note that this quadratic expression is always negative which is important since $p_c(t, \tau)$ is a probability, and any estimate of it must also be a probability.

We would expect the approximation (25) to not work well except when t and τ are both small.

Secondly, from [Weinberg and Kyprianou 2005], we have

$$p_c(t, \tau) = P(\aleph_2(\tau) \leq \aleph_1(t)), \quad (26)$$

where $(\aleph_1(t), \aleph_2(\tau)) \stackrel{d}{=} (\mathbf{Po}(t), \mathbf{Po}(\tau))$ are independent Poisson random variables. (Note that $\mathbf{Po}(\lambda)$ indicates a Poisson distribution with mean value λ).

A Monte Carlo estimator of (19) can be based on (26), by simulating from the distribution of \aleph_1 and evaluating cumulative probabilities of \aleph_2 . Specifically, one can use the estimator

$$\hat{p}_c(t, \tau) = \frac{1}{H} \sum_{j=1}^H P(\aleph_2(\tau) \leq \xi_j), \quad (27)$$

where each ξ_j is generated from a random variable with a $\mathbf{Po}(t)$ distribution. This can be used in conjunction with the estimator (23) to estimate the detection probability (20). It was found that an estimator based on (27) converged faster, and for less simulation runs, than an estimator based directly on the integral (19). Using (27) also provided better simulation estimates for the probability (20), rather than using the quadratic approximation (25).

Figure 7 in Appendix A is a simulation of (23), in the case where $N = 5$, $M = 3$, $l = 2$, $\xi_0 = 1$ and $\tau = 0.4$. The estimator's *Matlab* code can be found in Appendix B. The quadratic approximation (25) was used to estimate the probability (19). The j th simulation uses $K = 10^j$ iterations in the estimator (23). As can be observed, the Monte Carlo estimator (23) begins to settle down from the third simulation, which corresponds to $K = 1000$.

Figures 8-11 in Appendix A contain a number of simulations of the estimator (23), comparing the usage of the quadratic approximation (25) to the Poisson estimator (27). Figures 8 and 9 are for the case where $N = 7$, $M = 3$, $l = 2$, $\xi_0 = 0.001$ and $\tau = 1$. The j th simulation uses $K = 10^j$. Figure 8 compares an estimator using the quadratic approximation (25) to one using the Poisson estimator (27), with $H = 1000$. In this example, both estimators seem to be settling down after $K = 1000$ simulations. Figure 9 is for the same scenario, except only estimators based upon the Poisson estimate (27) are included. The three cases considered are for $H = K$, $H = 10$ and $H = 1000$. This simulation, as well as others investigated, showed that it is sufficient to take around $H = 1000$ to obtain a good Poisson estimate.

Figure 10 is a simulation for the case where $N = 5$, $M = 3$, $l = 2$, $\xi_0 = 1$ and $\tau = 5.3704$. As previously, $K = 10^j$ for the j th simulation. Three estimators are compared to an approximation based upon a numerical integration scheme¹. The numerical integration scheme gave a detection probability of 0.0095. The two Poisson estimators use $H = K$ and $H = 1000$ respectively. As can be observed, the Poisson based estimates coincide with the numerical value rather quickly, while the quadratic based estimator improves slowly.

¹This was provided by Mr Daniel Finch, EWRD, who used *Matlab's* numerical integration function *quad* to estimate the detection integral (20).

Figure 11 contains estimates for the case where $N = 20$, $M = 15$, $l = 2$, $\xi_0 = 1$ and $\tau = 1.4849$. The three estimators used are the same as in Figure 10. The numerical scheme gave a value of 0.1184. In this case, the Poisson estimator with $H = K$ has the best performance.

Other simulations considered showed that the estimator (23), when coupled with the quadratic approximation (25), had very poor performance when the threshold τ and the SNR ξ_0 were fairly large. In contrast to this, it was found that using the Poisson estimate (27) improved the estimation considerably, with reasonable results for $K = 1000$.

5 Conclusions and Future Directions

This report examined the Monte Carlo estimation of definite integrals. A number of estimators of the Gamma and Beta function were considered. These estimators performed reasonably well in practice. A number of estimators of the probability of detection, for a binary integration scheme, were also considered. These gave reasonable results in practice.

In future work, the detection probability estimator will be compared to other estimators, to gauge its performance.

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References

1. Billingsley, P (1986), Probability and Measure, John Wiley and Sons, 2nd Ed.
2. Bowman, Frank (1958), Introduction to Bessel Functions. Dover, New York.
3. Courant, R. and John, F. (1965), Introduction to Calculus and Analysis, Volume One. Wiley, New York.
4. Gerlach, K. (1999). New Results in Importance Sampling. IEEE Trans. Aero. Elec. Sys., Vol. 35, No. 3, pp. 917-925.
5. Kahn, H. (1950), Random Sampling (Monte Carlo) Techniques in Neutron Attenuation Problems-I. Nucleonics, pp. 27-37.
6. Levanon, N. (1988), Radar Principles. John Wiley & Sons, New York.
7. Mitchell, R.L. (1981), Importance Sampling applied to simulation of false alarm statistics, IEEE Trans. Aero. Elec. Sys. Vol AES-17, pp. 15-24.
8. Orsak, G. and Aazhang, B. (1989), On the Theory of Importance Sampling Applied to the Analysis of Detection Systems. IEEE Trans. Comms, Vol. 37, No. 4, pp. 332-339.
9. Ross, Sheldon (1983), Stochastic Processes. Wiley, New York.
10. Shnidman, D. (1995), Radar Detection Probabilities and their Calculations, IEEE Trans. Aero. Elec. Sys. Vol 31, pp. 928-950.
11. Shnidman, D. (1998), Binary integration for Swerling target fluctuations, IEEE Trans. Aero. Elec. Sys. Vol 34, pp. 1043-1053.
12. Tsypkin, A. G. and Tsypkin, G. G. (1988), Mathematical Formulas: Algebra, Geometry and Mathematical Analysis. Mir Publishers, Moscow.
13. Weinberg, G. V. (2004), Estimation of False Alarm Probabilities in Cell Averaging Constant False Alarm Rate Detectors via Monte Carlo Methods, DSTO-TR-1624.
14. Weinberg, G. V. and Kyprianou, R. (2005), Estimation of Detection Probabilities using Monte Carlo Sampling, to be submitted IEEE Trans. Aero. Elec. Sys.

Appendix A: Simulations

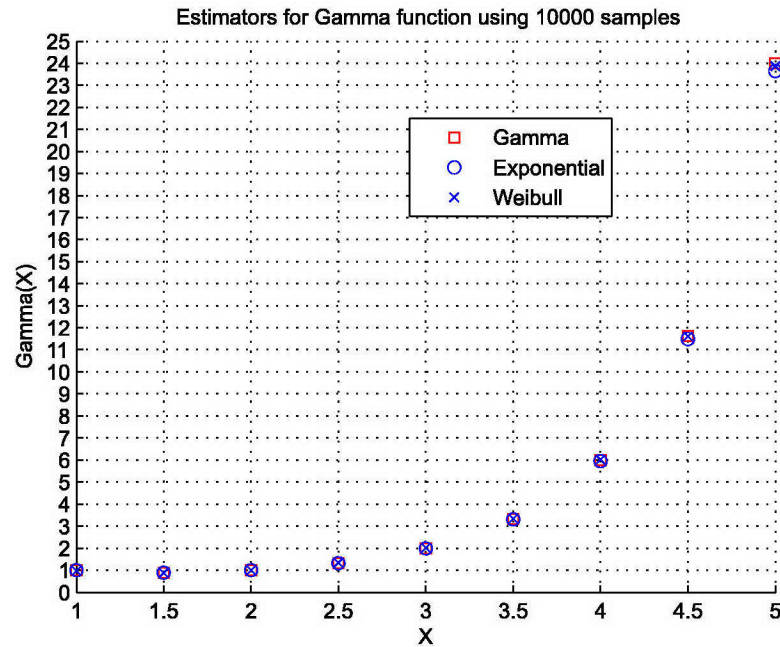


Figure 1: Simulations of the Gamma estimators (12) and (13), for a selection of values of X . In each case, both estimators are compared with an estimate based on the Matlab inbuilt Gamma function. The number of simulations used in each case is 10,000. In the legend, Gamma refers to the exact result, Exponential refers to the estimator (12) and Weibull refers to (13).

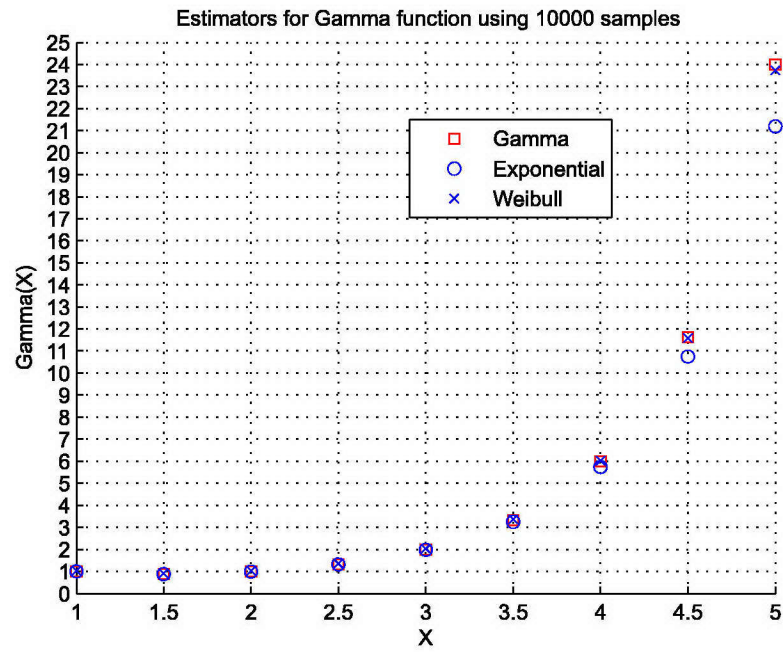


Figure 2: A simulation under exactly the same conditions as that of Figure 1, showing slightly worse results.

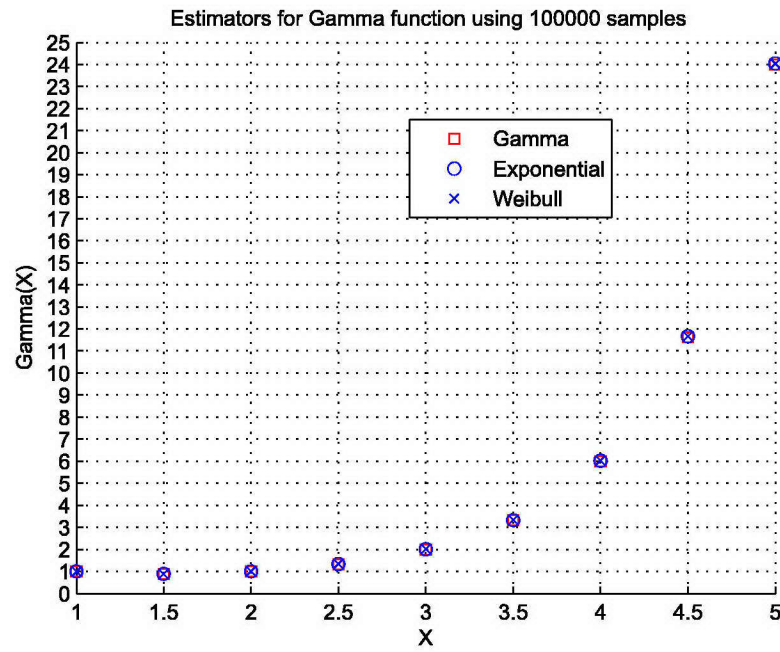


Figure 3: Simulations of Gamma function estimators as in Figures 1 and 2, except the number of simulation samples has been increased to 100,000.

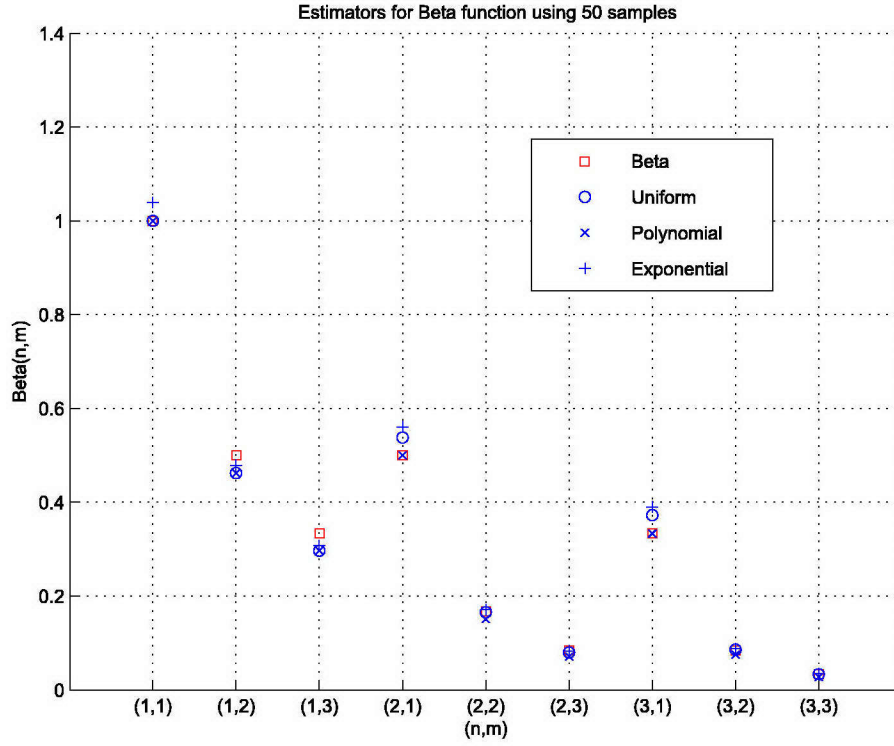


Figure 4: Simulations of the Beta integral estimators (14), (15) and (16). In each case, 50 simulations have been used to estimate the integral (11) with parameters (n, m) . In the legend, Beta refers to the exact result, while the other three refer to the estimators in order of appearance.

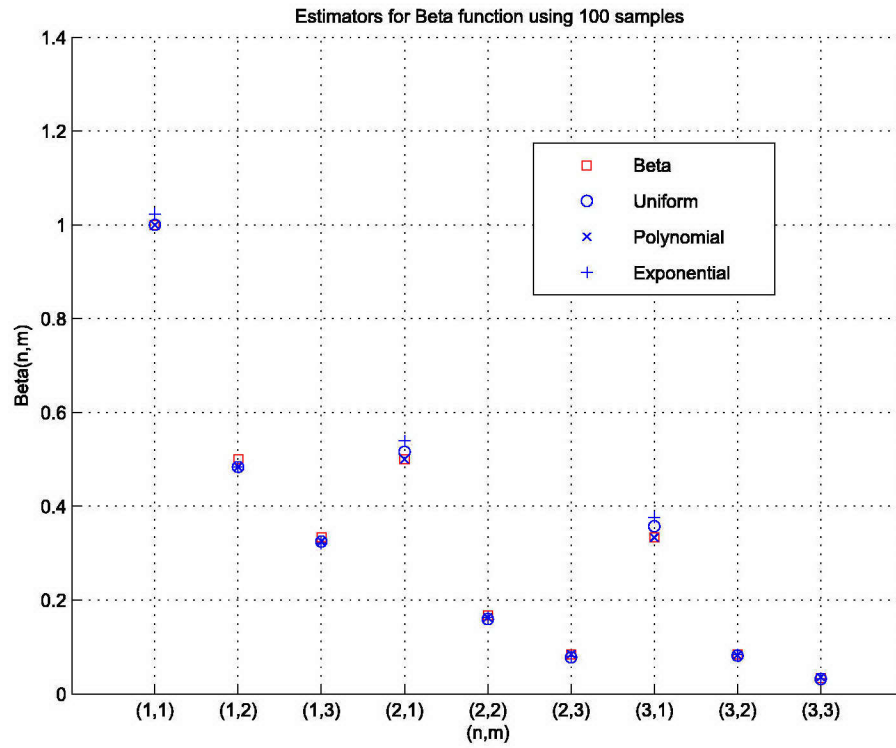


Figure 5: Same as for Figure 4, except 100 simulations are used to generate the estimates.

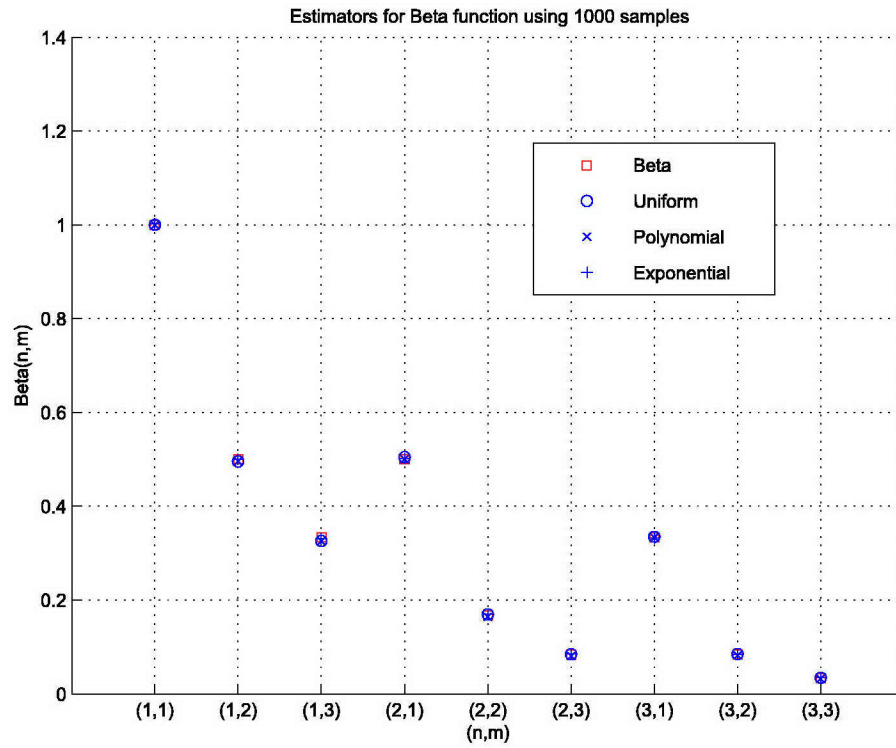


Figure 6: As for Figure 5, except 1000 simulations are used.

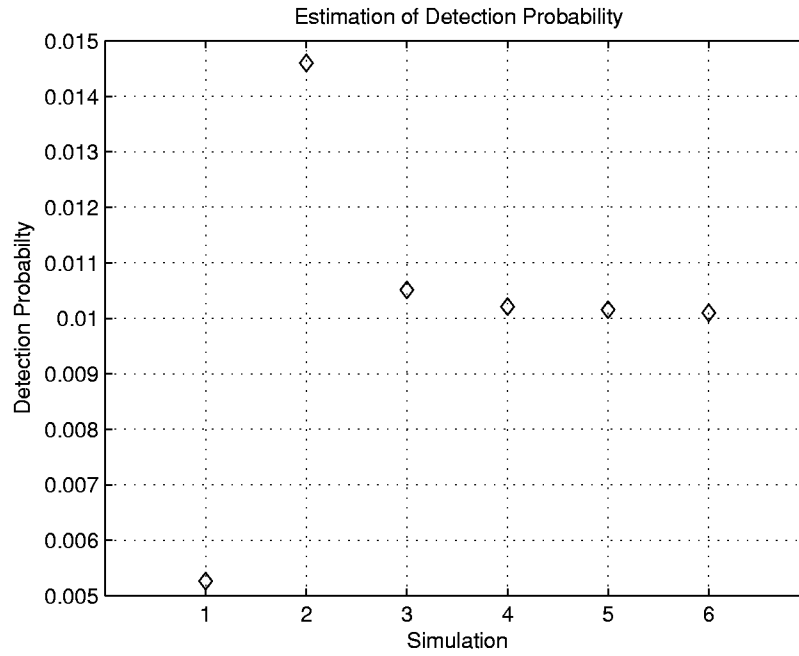


Figure 7: Simulation of (23) in the case of $N = 5$, $M = 3$, $l = 2$, $\xi_0 = 1$, $\tau = 0.4$ and for simulation j , $K = 10^j$. The quadratic approximation (25) was used to estimate the single pulse probability (19).

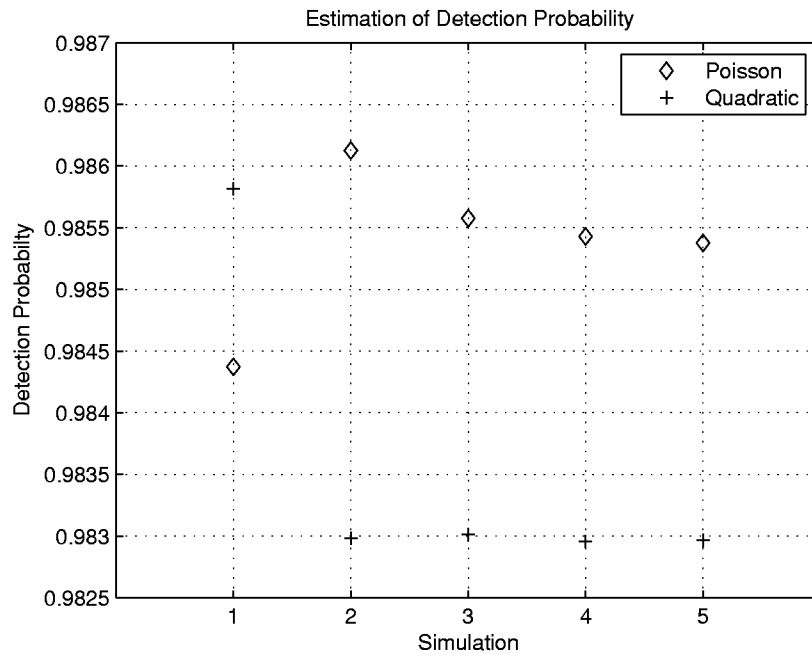


Figure 8: A comparison of estimator (23) using the quadratic approximation (25) and the Poisson estimator (27). In this case, $N = 7$, $M = 3$, $l = 2$, $\xi_0 = 0.001$, $\tau = 1$. For each j , $K = 10^j$, and $H = 1000$.

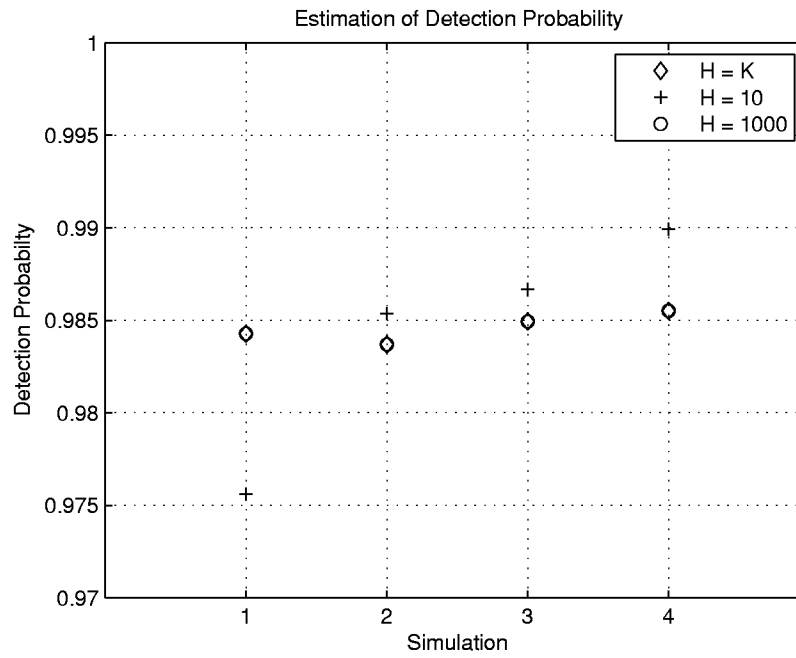


Figure 9: Comparison of three estimators using different Poissons estimators. The same parameter values are used as for the simulation of Figure 9, except the value of H is as shown.

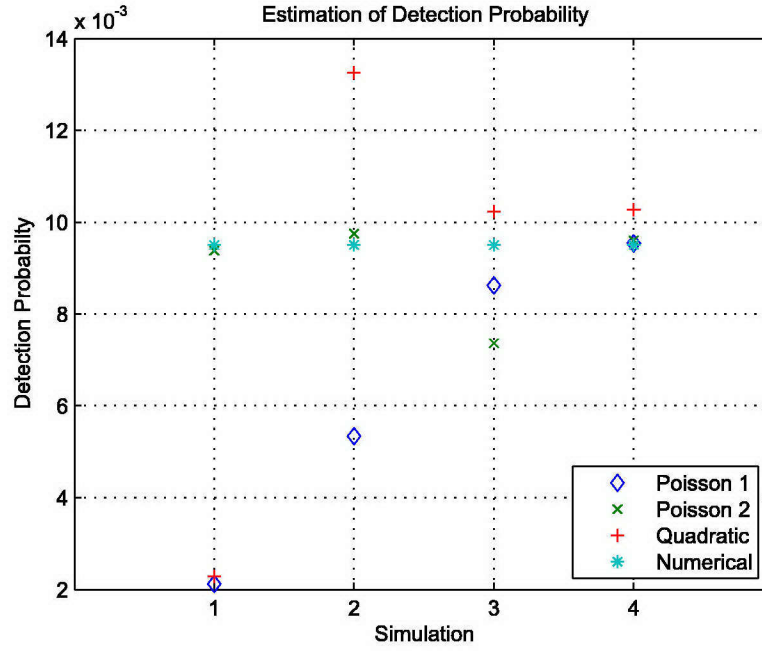


Figure 10: A simulation in the case where $N = 5$, $M = 3$, $l = 2$, $\xi_0 = 1$ and $\tau = 5.3704$. As in previous simulations, $K = 10^j$. The first Poisson estimate uses $H = K$, while the second uses $H = 1000$. The numerical estimate is based upon numerical integration applied directly to the detection probability integral (20).

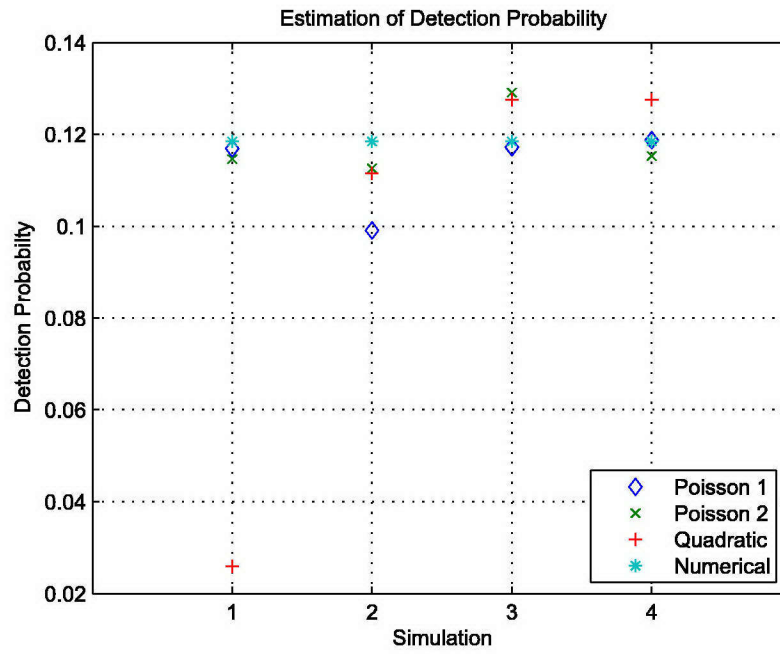
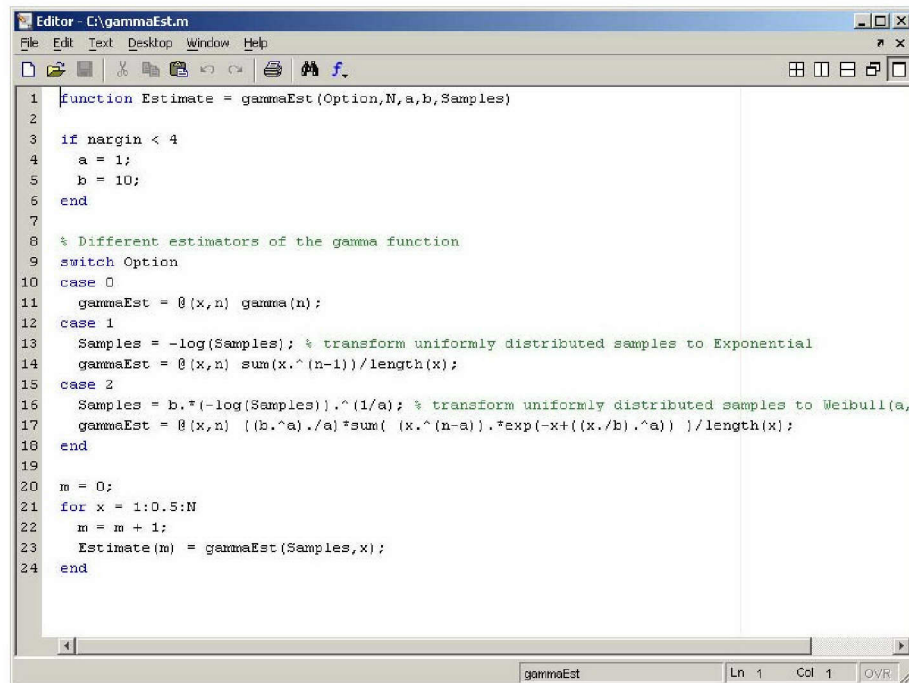


Figure 11: As for Figure 10, except $N = 20$, $M = 15$, $l = 2$, $\xi_0 = 1$ and $\tau = 1.4849$.

Appendix B: Matlab Code



```

1 function Estimate = gammaEst (Option,N,a,b,Samples)
2
3 if nargin < 4
4     a = 1;
5     b = 10;
6 end
7
8 % Different estimators of the gamma function
9 switch Option
10 case 0
11     gammaEst = @ (x,n) gamma(n);
12 case 1
13     Samples = -log(Samples); % transform uniformly distributed samples to Exponential
14     gammaEst = @ (x,n) sum(x.^(n-1))/length(x);
15 case 2
16     Samples = b.*(-log(Samples)).^(1/a); % transform uniformly distributed samples to Weibull(a,
17     gammaEst = @ (x,n) ((b.^a)./a)*sum( (x.^(n-a)).*exp(-x+((x./b).^a)) )/length(x);
18 end
19
20 m = 0;
21 for x = 1:0.5:N
22     m = m + 1;
23     Estimate(m) = gammaEst(Samples,x);
24 end

```

Figure 12: This function calculates Matlab's Gamma function (Option 0), estimator (12) (Option 1) and estimator (13) (Option 2). N Gamma values are calculated for integers and half-integers from 1 to N . Parameters a and b correspond to the α and β Gamma parameters. Samples is a vector of uniformly sampled points from the unit interval.


```

1 function Estimate = betaEst(Option,N,Samples,AsMatrix)
2
3 if nargin < 4
4     AsMatrix = false;
5 end
6
7 % Different estimators of the beta function
8 switch Option
9 case 0
10     distn = @(u,m) 1;
11     betaE = @(x,m,n) beta(m,n);
12 case 1
13     distn = @(u,m) u;
14     betaE = @(x,m,n) sum( (x.^(m-1)).*(1-x).^(n-1) )/length(x);
15 case 2
16     distn = @(u,m) u.^(1/m);
17     betaE = @(x,m,n) (1/m)*sum( ((1-x).^(n-1)) )/length(x);
18 case 3
19     distn = @(u,m) -log(1-((1-exp(-1))*u));
20     betaE = @(x,m,n) sum((1-exp(-1)).*exp(x).*(x.^(m-1)).*(1-x).^(n-1))/length(x);
21 end
22
23 Estimate = zeros(N,N);
24 for m = 1:N
25     for n = 1:N
26         Estimate(m,n) = betaE(distn(Samples,m),m,n);
27     end
28 end
29 if ~AsMatrix
30     Estimate = Estimate';
31     Estimate = Estimate(:);
32 end

```

Figure 13: Function for calculation of Matlab's Beta function (Option 0), estimator (14) (Option 1), estimator (15) (Option 2) and estimator (16) (Option 3). N is an integer indicating the size of the matrix of Beta values. $Samples$ is as for the Gamma implementation in Figure 12. $AsMatrix$, an optional argument, provides a better formatted output.

```

1 function Estimate = binintEst(Option, K1, K2, N, M, l, x0, tau)
2
3 Sum = 0;
4 for j=1:K1
5     t = gamrnd(1, x0/l);
6     if Option == 1
7         Prob = sum(poisscdf(poissrnd(t,K2,1),tau))/K2;
8     else
9         Prob = (exp(-tau))*(1 + tau*t - 0.5*(tau-0.5*tau^2)*t^2);
10    end
11
12    binSum = 0;
13    for i =M:N
14        binSum = binSum + binopdf(i, N, Prob);
15    end
16    Sum = Sum + binSum;
17 end
18 Estimate = Sum/K1;

```

Figure 14: Implementation of the binary integration detection probability estimator (23). Option enables the single pulse probability of detection to be estimated via (25) (Option 0) or (27) (Option 1). Parameter K1 is the number of Monte Carlo simulations for the main estimator (23), while K2 is for the Monte Carlo estimator (27). N, M, l and tau are the corresponding parameters in the binary integration scheme, while $x_0 = \xi_0$.

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Graham V. Weinberg and Ross Kyprianou

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19. ABSTRACT The approximation of definite integrals using Monte Carlo simulations is the focus of the work presented here. The general methodology of estimation by sampling is introduced, and is applied to the approximation of two special functions of mathematics: the Gamma and Beta functions. A significant application, in the context of radar detection theory, is based upon the work of [Shnidman 1998]. The latter considers problems associated with the optimal choice of binary integration parameters. We apply the techniques of Monte Carlo simulation to estimate binary integration detection probabilities.					